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# HARMONIC MAPS OF THE TWO-SPHERE INTO THE COMPLEX HYPERQUADRIC

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## Introduction

Let $G(k, n; \mathbf{C})$ denote the Grassmann manifold of all $k$-dimensional subspaces $\mathbf{C}^k$ of complex $n$-space $\mathbf{C}^n$. Let $P_{n-1}$ denote complex projective $(n-1)$ space, $P_{n-1} = G(1, n; \mathbf{C})$ and let $Q_{n-2} \subset P_{n-1}$ denote the complex hyperquadric, that is, the complex hypersurface, of $P_{n-1}$ defined by the equation

$$Z_0^2 + Z_1^2 + \cdots + Z_{n-1}^2 = 0,$$

where $\{Z_0, \cdots, Z_{n-1}\}$ are homogeneous coordinates of $P_{n-1}$. $Q_{n-2}$ has a natural Kähler metric which it inherits as a complex submanifold of $P_{n-1}$. In this note we will study the minimal immersions or harmonic maps of the two-sphere $S^2$ into $Q_{n-2}$. Our result can be described as follows: To each harmonic map $f\colon S^2 \to Q_{n-2}$ we associate a *directrix curve* $\Delta_f$: $S^2 \to G(2, n; \mathbf{C})$ which is either a holomorphic curve or a degenerate harmonic map. (The degenerate harmonic maps arise in the study of harmonic maps $S^2 \to G(2, n; \mathbf{C})$. In [4] it is shown that they can be constructed from holomorphic curves $S^2 \to P_{n-1}$.) The directrix curve $\Delta_f$ will be shown to satisfy strong nullity conditions, in the sense that its $l$th osculating space is null for $0 \leqslant l \leqslant r$ (where $r \geqslant 0$ depends on $f$). The harmonic map $f$ can be recovered from its directrix curve $\Delta_f$ via differentiation and the choice of holomorphic sections of $P_1$ bundles over $S^2$. This description and Calabi's description of minimal maps $S^2 \to S^N$ [1] are related. In fact, the nullity conditions on the directrix curves of harmonic maps $S^2 \to Q_{n-2}$ are similar to those on the directrix curves of minimal maps $S^2 \to S^N$.

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In §1 we will discuss the geometry of the spaces $G(k, n; \mathbf{C})$, $G(2, n; \mathbf{R})$, and $Q_{n-2}$. In §2 we will give an account of the basic results on harmonic maps $M^2 \to G(2, n; \mathbf{C})$ as developed in [4]. We will omit proofs and refer the reader to [3] or [4].

## 1. Some geometry

Let $V, W \in \mathbf{C}^n$,

$$V = (v_1, \cdots, v_n), \qquad W = (w_1, \cdots, w_n). \tag{1.1}$$

We equip $\mathbf{C}^n$ with two inner products. First, with the standard Hermitian inner product so that

$$\langle V, W \rangle = \sum v_A \overline{w}_A = \sum v_A w_{\overline{A}} \tag{1.2}$$

and, second, with the symmetric inner product so that

$$(V, W) = \sum v_A w_A. \tag{1.3}$$

Of course $\langle V, \overline{W} \rangle = (V, W)$.

A frame consists of an ordered set of $n$ linearly independent vectors $Z_A$, so that

$$Z_1 \wedge \cdots \wedge Z_n \neq 0. \tag{1.4}$$

It is called unitary if

$$\langle Z_A, Z_B \rangle = \delta_{A\overline{B}}. \tag{1.5}$$

The space of unitary frames can be identified with the unitary group $U(n)$. Writing

$$dZ_A = \sum_B \omega_{A\overline{B}} Z_B, \tag{1.6}$$

the $\omega_{A\overline{B}}$ are the Maurer-Cartan forms of $U(n)$. They are skew-Hermitian, i.e. we have

$$\omega_{A\overline{B}} + \overline{\omega}_{B\overline{A}} = 0. \tag{1.7}$$

Taking the exterior derivative of (1.6), we get the Maurer-Cartan equations of $U(n)$:

$$d\omega_{A\overline{B}} = \sum_C \omega_{A\overline{C}} \wedge \omega_{C\overline{B}}. \tag{1.8}$$

We call equations (1.6) and (1.8) the *structure equations* of the frame.

An element of $G(k, n; \mathbf{C})$ can be given by a multivector $Z_1 \wedge \cdots \wedge Z_k \neq 0$, defined up to a factor. The vectors $Z_\alpha$, $i \leqslant \alpha \leqslant k$, and their orthogonal vectors $Z_i$, $k + 1 \leqslant i \leqslant n$, are defined up to a transformation of $U(k)$ and $U(n - k)$, respectively. Thus, the form

$$ds^2 = \sum_{\alpha,i} \omega_{\alpha \bar{i}} \bar{\omega}_{\alpha \bar{i}} \tag{1.9}$$

is a positive definite Hermitian form on $G(k, n; \mathbf{C})$ and defines a Hermitian metric. Its Kähler form is

$$\Omega = \frac{\sqrt{-1}}{2} \sum_{\alpha,i} \omega_{\alpha \bar{i}} \wedge \bar{\omega}_{\alpha \bar{i}}. \tag{1.10}$$

By using (1.8) it can be verified immediately that $\Omega$ is closed, so that the metric $ds^2$ is Kählerian. When $k = 1$, $G(1, n; \mathbf{C})$ is complex projective $(n-1)$-space $P_{n-1}$ and the metric (1.9) is called the Fubini-Study metric.

It is easy to see that complex conjugation is an isometry of $G(k, n; \mathbf{C})$ with the metric (1.9). Its fixed point set is the real Grassmann manifold $G(k, n; \mathbf{R})$. Thus $G(k, n; \mathbf{R})$ lies totally geodesically in $G(k, n; \mathbf{C})$.

An element $S$ of $G(k, n; \mathbf{C})$ is called null if

$$(V, W) = 0 \quad \text{for all vectors } V, W \in S$$

(equivalently $\langle V, \overline{W} \rangle = 0$). In particular an element $L \in P_{n-1}$ is null if $(Z, Z) = 0$ for any $Z \in L$. The manifold of all null lines is a complex hypersurface of $P_{n-1}$ called the complex hyperquadric and denoted $Q_{n-2}$. The Fubini-Study metric induces a Kähler metric on $Q_{n-2}$. If $Z$ is a homogeneous coordinate vector for a point on $Q_{n-2}$, then $(Z, Z) = 0$, so $\langle Z, \overline{Z} \rangle = 0$. That is, $Z$ is orthogonal to $\overline{Z}$. Define a map

$$\phi \colon Q_{n-2} \to G(2, n; \mathbf{R})$$

as follows: Represent a point $p \in Q_{n-2}$ by a homogeneous coordinate vector $Z$ and set

$$\phi(p) = \frac{\sqrt{-1}}{2} Z \wedge \overline{Z};$$

$\phi$ is clearly well defined. $\phi$ is one-to-one and onto. It follows easily using (1.6) and (1.9) that $\phi$ is an isometry. Using $\phi$ we will henceforth identify $Q_{n-2}$ and $G(2, n; \mathbf{R})$ (for more details see [2]).

## 2. Harmonic maps of surfaces into $G(2, n; \mathbf{C})$

Let $M$ be an oriented Riemannian surface and let $f \colon M \to G(2, n; \mathbf{C})$ be a nonconstant harmonic map. Denote the Riemannian metric on $M$ by $ds_M^2 = \phi \cdot \bar{\phi}$, where $\phi$ is a $(1,0)$ form on $M$. For $x \in M$ the image $f(x) \in G(2, n; \mathbf{C})$ has an orthogonal space $f(x)^{\perp} \in G(n-2, n; \mathbf{C})$. If $Z \in f(x)$, we can write

$$dZ \equiv X\phi + Y\bar{\phi} \mod f(x), \tag{2.1}$$

where $X, Y \in f(x)^{\perp}$. If $Z \in \mathbf{C}^n - \{0\}$, we denote by $[Z]$ the point in $P_{n-1}$ with $Z$ as the homogeneous coordinate vector. Then

$$\partial\colon [Z] \to [X] \quad \text{and} \quad \bar{\partial}\colon [Z] \to [Y], \tag{2.2}$$

if not zero, are well-defined projective collineations of the projectivized space $[f(x)]$ into $[f(x)^{\perp}]$. These maps are called the *fundamental collineations*. The fundamental collineation $\bar{\partial}$ (resp. $\partial$) is zero if and only if $f$ is holomorphic (resp. antiholomorphic).

Choose a unitary frame $Z_A$ so that $\{Z_1, Z_2\}$ span $f(x)$. By (1.9) the one-forms $\omega_{\alpha\bar{i}}$, $\alpha = 1, 2$, $i = 3, \cdots, n$, form a unitary coframing of $G(2, n; \mathbf{C})$. We have

$$f^*\omega_{\alpha\bar{i}} = a_{\alpha\bar{i}}\phi + b_{\alpha\bar{i}}\bar{\phi}. \tag{2.3}$$

Set

$$\begin{aligned} Da_{\alpha\bar{i}} &= da_{\alpha\bar{i}} - a_{\beta\bar{i}}\omega_{\alpha\bar{\beta}} + a_{\alpha\bar{j}}\omega_{j\bar{i}} - \sqrt{-1}\,a_{\alpha\bar{i}}\eta, \\ Db_{\alpha\bar{i}} &= db_{\alpha\bar{i}} - b_{\beta\bar{i}}\omega_{\alpha\bar{\beta}} + b_{\alpha\bar{j}}\omega_{j\bar{i}} + \sqrt{-1}\,b_{\alpha\bar{i}}\eta, \end{aligned} \tag{2.4}$$

where $\eta$ is the connection one-form of the metric $ds^2_M$.

**Theorem 2.1.** *The property that $f$ is a harmonic map is expressed by one of the following equivalent conditions*:

(a) $Da_{\alpha\bar{i}} \equiv 0 \bmod \phi$, *or*

(b) $Db_{\alpha\bar{i}} \equiv 0 \bmod \bar{\phi}$.

Throughout this paper the criterion of Theorem 2.1 will be used repeatedly.

It follows from the harmonicity of $f$ that the fundamental collineations have constant rank, except perhaps at isolated points. Denoting $\dim \partial[f(x)] = k_1 - 1$, we define the $\partial$-transform of $f$,

$$\partial f\colon M \to G(k_1, n; \mathbf{C}), \tag{2.5}$$

by $(\partial f)(x) = \partial[f(x)]$, $x \in M$. Similarly we define the $\bar{\partial}$-transform of $f$.

**Theorem 2.2.** *Let $f\colon M \to G(2, n; \mathbf{C})$ be a harmonic map. Then*

(a) *The maps $\partial f$, $\bar{\partial} f$ are harmonic.*

(b) *If $k_1 = 2$, $\bar{\partial}\partial f$ is $f$ itself.*

Repeating the construction of the theorem we get two sequences of harmonic maps

$$\begin{aligned} &L_0(=f) \xrightarrow{\partial} L_1 \xrightarrow{\partial} L_2 \to \cdots, \\ &L_0 \xrightarrow{\bar{\partial}} L_{-1} \xrightarrow{\bar{\partial}} L_{-2} \to \cdots \end{aligned} \tag{2.6}$$

whose image spaces are connected by fundamental collineations. Such sequences are called *harmonic sequences*. When all the $L_p$'s are two-dimensional we can combine the sequence into one:

$$\cdots L_{-2} \underset{\bar\partial}{\overset{\partial}{\rightleftarrows}} L_{-1} \underset{\bar\partial}{\overset{\partial}{\rightleftarrows}} L_0 \underset{\bar\partial}{\overset{\partial}{\rightleftarrows}} L_1 \cdots . \tag{2.7}$$

Two consecutive spaces $L_p(x)$ and $L_{p+1}(x)$, $x \in M$, of a harmonic sequence are orthogonal. If any two members of a harmonic sequence are orthogonal, then the sequence is called a *Frenet harmonic sequence*. A Frenet harmonic sequence whose members span the ambient space is called a *full Frenet harmonic sequence*.

Because the two-sphere has no nonzero holomorphic differential forms, we have

**Theorem 2.3.** *Let $L_\lambda$: $S^2 \to G(2, n;\ \mathbf{C})$, $0 \leqslant \lambda \leqslant s-1$, $n \geqslant 2s$, be harmonic maps which form a Frenet harmonic sequence*

$$L_0 \underset{\bar\partial}{\overset{\partial}{\rightleftarrows}} L_1 \underset{\bar\partial}{\overset{\partial}{\rightleftarrows}} \cdots \underset{\bar\partial}{\overset{\partial}{\rightleftarrows}} L_{s-1}. \tag{2.8}$$

*Let $\pi_+$: $L_{s-1}^{\perp} \to L_0$ and $\pi_-$: $L_0^{\perp} \to L_{s-1}$ be the orthogonal projections. Consider the maps*

$$\begin{aligned} D_+ &= \underbrace{\partial \circ \cdots \circ \partial}_{s-1} \circ (\pi_+ \circ \partial) \colon L_{s-1} \to L_{s-1}, \\ D_- &= \underbrace{\bar\partial \circ \cdots \circ \bar\partial}_{s-1} \circ (\pi_- \circ \bar\partial) \colon L_0 \to L_0. \end{aligned} \tag{2.9}$$

*The trace and determinant of $D_+$ and $D_-$ vanish identically. In particular the maps $\pi_+ \circ \partial$: $L_{s-1} \to L_0$ and $\pi_- \circ \partial$: $L_0 \to L_{s-1}$ are degenerate. When $n = 2s$, $\partial$: $L_{s-1} \to L_0$ and $\bar\partial$: $L_0 \to L_{s-1}$; so in this case the fundamental collineations are degenerate.*

Suppose that (2.8) is a full Frenet harmonic sequence (so that $n = 2s$). By Theorem 2.3 the fundamental collineation $\bar\partial$: $L_0 \to L_{s-1}$ is zero or has rank one. If the former, then $L_0$ is a holomorphic curve $S^2 \to G(2, n;\ \mathbf{C})$. If the latter, then by Theorem 2.2 the image of $\bar\partial$ describes a harmonic map $S^2 \to G(1, n;\ \mathbf{C}) = P_{n-1}$. The Din-Zakrzewski description of harmonic maps $S^2 \to P_{n-1}$ [5] gives that the image of $\bar\partial$ is an element of the classical Frenet frame of some holomorphic curve $S^2 \to P_{n-1}$. Thus we see that full Frenet harmonic sequences lead to holomorphic curves. The idea of [4] is to exploit this by associating to a given harmonic sequence a new harmonic sequence which is full and Frenet. This association is effected by an inductive construction called *crossing*. Crossing associates a Frenet harmonic sequence of length $l+1$ to a given Frenet harmonic sequence of length $l$.

Now suppose that $L_0\colon S^2 \to G(2,n;\ \mathbf{R}) \subseteq G(2,n;\ \mathbf{C})$ is harmonic, so in particular $L_0 = \overline{L}_0$. In fact the harmonic sequences of (2.6) are, in this case, conjugates of one another. If all the $L_i$'s are projective lines we have the harmonic sequence

$$(2.10) \qquad L_{-s} \xrightarrow{\partial} L_{-(s-1)} \xrightarrow{\partial} \cdots \xrightarrow{\partial} L_{-1} \xrightarrow{\partial} L_0 \xrightarrow{\partial} L_1 \to \cdots \xrightarrow{\partial} L_s,$$

where each $L_\rho$, $-s \leqslant \rho \leqslant s$, is the image of the previous one under a fundamental collineation and where $L_{-\rho} = \overline{L}_\rho$, $-s \leqslant \rho \leqslant s$. Such a harmonic sequence will be called a *real harmonic sequence*. A harmonic sequence of the form (2.10) with $L_0$ deleted and satisfying $L_{-\rho} = \overline{L}_\rho$ will also be called a *real harmonic sequence*. In §3 we will show that the construction of crossing takes real harmonic sequences to real harmonic sequences.

## 3. Crossing

Consider the real Frenet harmonic sequence

$$(3.1) \qquad L_{-s} \xrightarrow{\partial} \cdots \xrightarrow{\partial} L_{-1} \xrightarrow{\partial} L_0 \xrightarrow{\partial} L_1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} L_s.$$

Denote the fundamental collineation $\partial\colon L_\rho \to L_{\rho+1}$ (resp. $\overline{\partial}\colon L_\rho \to L_{\rho-1}$) by $\partial_\rho$ (resp. $\overline{\partial}_\rho$) for $-s \leqslant \rho \leqslant s$. Then because (3.1) is a real harmonic sequence

$$(3.2) \qquad \sigma(\partial_\rho) = \overline{\partial}_{-\rho}, \qquad \sigma(\overline{\partial}_\rho) = \partial_{-\rho},$$

where $\sigma$ denotes complex conjugation. By $\sigma(\partial_\rho) = \overline{\partial}_{-\rho}$ we mean that if $V \in L_\rho$, then

$$\overline{\partial_\rho(V)} = \overline{\partial}_{-\rho}(\overline{V}).$$

This follows immediately from (2.1) and its conjugate. Because the elements of (3.1) are mutually orthogonal

$$(3.3) \qquad \overline{\partial}_\rho = -\sigma\,{}^t(\partial_{\rho-1}),$$

where ${}^t$ denotes the adjoint with respect to the symmetric inner product.

**Theorem 3.1.** *The sequence obtained from* (3.1) *by adding* $\partial_s\colon L_s \to L_{s+1}$ (*or by adding* $\overline{\partial}_{-s}\colon L_{-s} \to L_{-(s+1)}$) *is a Frenet harmonic sequence.*

*Proof.* The structure equations associated to (3.1) show that $L_{s+1}$ is orthogonal to $L_{-(s-1)}, \cdots, L_0, \cdots, L_s$ and that $L_{-(s+1)}$ is orthogonal to $L_{-s}, \cdots, L_0, \cdots, L_{s-1}$. Let $\pi_+\colon L_{s+1} \to L_{-s}$ and $\pi_-\colon L_{-(s+1)} \to L_s$ denote the orthogonal projections. Consider the maps $\pi_+ \circ \partial_s\colon L_s \to L_{-s}$ and $\pi_- \circ \overline{\partial}_{-s}\colon L_{-s} \to L_s$. Clearly

$$(3.4) \qquad \pi_+ \circ \partial_s = \sigma(\pi_- \circ \overline{\partial}_{-s}).$$

From the structure equations it follows that

$$\pi_+ \circ \partial_s = -\sigma\,{}^t(\pi_- \circ \bar\partial_{-s}). \tag{3.5}$$

Thus,

$$\pi_+ \circ \partial_s = -{}^t(\pi_+ \circ \partial_s). \tag{3.6}$$

By Theorem 2.3 the map $L_s \to L_s$ given by the composition $\partial_{s-1} \circ \cdots \circ \partial_{-s} \circ (\pi_+ \circ \partial_s)$ is degenerate. By assumption however the maps $\partial_\rho$: $L_\rho \to L_{\rho+1}$, $-s \leqslant \rho \leqslant s$, are nondegenerate and so $\det(\pi_+ \circ \partial_s) = 0$. This together with (3.6) implies that $\pi_+ \circ \partial_s = 0$ and therefore $L_{s+1}$ is orthogonal to $L_{-s}$. q.e.d.

Consider the real Frenet harmonic sequence

$$L_{-s} \overset{\partial_{-s}}{\to} \cdots \overset{\partial_{-2}}{\to} L_{-1} \overset{\partial_{-1}}{\to} L_1 \overset{\partial_1}{\to} \cdots \overset{\partial_{s-1}}{\to} L_s. \tag{3.7}$$

The proof of Theorem 3.1 gives

**Theorem 3.2.** *The sequence obtained from* (3.7) *by adding* $\partial_s$: $L_s \to L_{s+1}$ (*or by adding* $\bar\partial_{-s}$: $L_{-s} \to L_{-(s+1)}$) *is a Frenet harmonic sequence.*

Consider the Frenet harmonic sequence

$$L_{-s} \overset{\partial_{-s}}{\to} \cdots \overset{\partial_{-1}}{\to} L_0 \overset{\partial_0}{\to} \cdots \to L_s \overset{\partial_s}{\to} L_{s+1} \tag{3.8}$$

obtained from (3.1) by adding $\partial_s$: $L_s \to L_{s+1}$. Suppose that the harmonic sequence obtained from (3.8) by adding $\bar\partial_{-s}$: $L_{-s} \to L_{-(s+1)}$ is not Frenet, i.e., $L_{-(s+1)} = \bar L_{s+1}$ is not orthogonal to $L_{s+1}$. Let $\pi$: $L_{-(s+1)} \to L_{s+1}$ denote the orthogonal projection and consider the map $\pi \circ \bar\partial_{-s}$: $L_{-s} \to L_{s+1}$. By Theorem 2.3 $\pi \circ \bar\partial_{-s}$ has rank one. Let $Z_{2s+2} \in L_{s+1}$ be a vector of unit length whose span is the image of $\pi \circ \bar\partial_{-s}$ and choose $Z_{2s+1} \in L_{s+1}$ so that $\{Z_{2s+1}, Z_{2s+2}\}$ is a unitary framing of $L_{s+1}$. Now inductively choose a unitary framing $\{Z_{2\rho-1}, Z_{2\rho}\}$ of $L_\rho$, $\rho > 0$, by setting

$$\partial_\rho(Z_{2\rho-1}) = \alpha_{2\rho+1} Z_{2\rho+1} \tag{3.9}$$

(where $\alpha_{2\rho+1}$ is a scalar) and choosing $Z_{2\rho}$ orthogonal to $Z_{2\rho-1}$ in $L_\rho$. With respect to the bases $\{Z_{2\rho-1}, Z_{2\rho}\}$ and $\{Z_{2\rho+1}, Z_{2\rho+2}\}$ of $L_\rho$ and $L_{\rho+1}$, respectively, the matrix $A_\rho$ of $\partial_\rho$ has the form

$$A_\rho = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}, \quad 1 \leqslant \rho \leqslant s. \tag{3.10}$$

The conjugated frames $\{\bar Z_{2\rho-1}, \bar Z_{2\rho}\}$ are unitary frames of the spaces $\bar L_\rho = L_{-\rho}$. Let $A_{-\rho}$ denote the matrix representation of $\partial_{-\rho}$: $L_{-\rho} \to L_{-\rho+1}$. Then with respect to the bases $\{\bar Z_{2\rho}, \bar Z_{2\rho-1}\}$ and $\{\bar Z_{2\rho-2}, \bar Z_{2\rho-3}\}$ of $L_{-\rho}$ and $L_{-\rho+1}$, respectively, $A_{-\rho}$ has the form

$$A_{-\rho} = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}, \quad -2 \geqslant -\rho \geqslant -s. \tag{3.11}$$

(Note the ordering of the frames.) (3.11) follows from (3.2), (3.3), and (3.9).

Since $Z_{2s+1}$ is orthogonal to the image of $\pi \circ \bar{\partial}_{-s}$, $Z_{2s+1}$ is orthogonal to $L_{-(s+1)}$. In particular $Z_{2s+1}$ is orthogonal to $\bar{Z}_{2s+1}$. Let $C$ denote the matrix representation of $\pi \circ \bar{\partial}_{-s}$ with respect to the bases $\{\bar{Z}_{2s}, \bar{Z}_{2s-1}\}$ and $\{Z_{2s+1}, Z_{2s+2}\}$ of $L_{-s}$ and $L_{s+1}$, respectively. Then since $\partial_s(Z_{2s-1}) = \alpha_{2s+1}Z_{2s+1}$, it follows that

$$\bar{\partial}_{-s}(\bar{Z}_{2s-1}) = \bar{\alpha}_{2s+1}\bar{Z}_{2s+1}.$$

Thus, $\pi \circ \bar{\partial}_{-s}(\bar{Z}_{2s-1}) = 0$. $C$ has the form

$$C = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}. \tag{3.12}$$

We need to choose a framing of $L_0$. Let $Z_0 \in L_0$ be a vector of unit length satisfying

$$\partial_{-1}(\bar{Z}_2) = \bar{\alpha}_0\bar{Z}_0. \tag{3.13}$$

Now (3.3) and (3.9) imply that

$$\bar{\partial}_\rho(Z_{2\rho}) = (\ )Z_{2\rho-2} \tag{3.14}$$

and (3.13) implies that $\bar{\partial}_1(Z_2) = \alpha_0 Z_0$. So

$$\bar{\partial}_1 \circ \bar{\partial}_2 \circ \cdots \circ \bar{\partial}_s \circ (\pi \circ \bar{\partial}_{-s})(\bar{Z}_{2s}) = (\ )Z_0.$$

(3.11) and (3.13) imply that

$$\partial_{-1} \circ \cdots \circ \partial_{-s}(\bar{Z}_{2s}) = (\ )\bar{Z}_0.$$

We claim that $\langle \bar{Z}_0, Z_0 \rangle = 0$. To see this note that

$$\begin{aligned} &\langle \partial_{-1} \circ \cdots \circ \partial_{-s}(\bar{Z}_{2s}), \bar{\partial}_1 \circ \cdots \circ \bar{\partial}_s \circ (\pi \circ \bar{\partial}_{-s})(\bar{Z}_{2s}) \rangle \\ &= \langle \bar{Z}_{2s}, \bar{\partial}_{-(s-1)} \circ \cdots \circ \bar{\partial}_0 \circ \bar{\partial}_1 \circ \cdots \circ \bar{\partial}_s \circ (\pi \circ \bar{\partial}_{-s})(\bar{Z}_{2s}) \rangle \\ &= \langle \bar{Z}_{2s}, D(\bar{Z}_{2s}) \rangle, \end{aligned} \tag{3.15}$$

where $D = \bar{\partial}_{-(s-1)} \circ \cdots \circ \bar{\partial}_s \circ (\pi \circ \bar{\partial}_{-s})$.

By Theorem 2.3 $D$ has zero trace. As $D(\bar{Z}_{2s-1}) = 0$, $\langle \bar{Z}_{2s}, D(\bar{Z}_{2s}) \rangle = 0$, and therefore $\langle \bar{Z}_0, Z_0 \rangle = 0$. Thus $\{\bar{Z}_0, Z_0\}$ is a unitary framing of $L_0$.

Define the projective lines

$$\lambda_\rho = Z_{2\rho-2} \wedge Z_{2\rho-1}, \quad \lambda_{-\rho} = \bar{\lambda}_\rho, \qquad 1 \leqslant \rho \leqslant s+1. \tag{3.16}$$

**Theorem 3.3.** *The projective lines* $\lambda_{-\rho}$ *and* $\lambda_\rho$, $1 \leqslant \rho \leqslant s+1$, *form a Frenet harmonic sequence*

$$\lambda_{-(s+1)} \xrightarrow{\partial} \lambda_{-s} \xrightarrow{\partial} \cdots \to \lambda_{-1} \xrightarrow{\partial} \lambda_1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} \lambda_{s+1}. \tag{3.17}$$

*Proof.* Let $A_{-1}$ (resp. $A_0$) denote the matrix representative of $\partial_{-1}$ (resp. $\partial_0$) with respect to the bases $\{\bar{Z}_2, \bar{Z}_1\}$ and $\{\bar{Z}_0, Z_0\}$ of $L_{-1}$ and $L_0$ (resp. $\{\bar{Z}_0, Z_0\}$ and $\{Z_1, Z_2\}$ of $L_0$ and $L_1$). Then $A_i$, $i = 0, -1$, has the form

$$A_i = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}.$$

Hence

$$A_\rho = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}, \qquad -s \leqslant \rho \leqslant s. \tag{3.18}$$

We have

$$d\begin{pmatrix} Z_{2\rho-1} \\ Z_{2\rho} \end{pmatrix} = -\,{}^t\bar{A}_{\rho-1}\bar{\phi}\begin{pmatrix} Z_{2\rho-3} \\ Z_{2\rho-2} \end{pmatrix} + \pi_\rho\begin{pmatrix} Z_{2\rho-1} \\ Z_{2\rho} \end{pmatrix} + A_\rho\phi\begin{pmatrix} Z_{2\rho+1} \\ Z_{2\rho+2} \end{pmatrix} \quad \text{for } 1 \leqslant \rho \leqslant s,$$

$$d\begin{pmatrix} \bar{Z}_0 \\ Z_0 \end{pmatrix} = -\,{}^t\bar{A}_{-1}\bar{\phi}\begin{pmatrix} \bar{Z}_2 \\ \bar{Z}_1 \end{pmatrix} + \pi_0\begin{pmatrix} \bar{Z}_0 \\ Z_0 \end{pmatrix} + A_0\phi\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}, \tag{3.19}$$

$$d\begin{pmatrix} \bar{Z}_{2\rho} \\ \bar{Z}_{2\rho-1} \end{pmatrix} = -\,{}^t\bar{A}_{-(\rho+1)}\bar{\phi}\begin{pmatrix} \bar{Z}_{2\rho+2} \\ \bar{Z}_{2\rho+1} \end{pmatrix} + \pi_{-\rho}\begin{pmatrix} \bar{Z}_{2\rho} \\ \bar{Z}_{2\rho-1} \end{pmatrix} + A_{-\rho}\phi\begin{pmatrix} \bar{Z}_{2\rho-2} \\ \bar{Z}_{2\rho-3} \end{pmatrix} \quad \text{for } 1 \leqslant \rho \leqslant s-1.$$

The structure equations and the harmonicity of the sequence (3.8) imply that the matrices of 1-forms $\pi_\rho$ have the form

$$\pi_\rho = \begin{pmatrix} - & q_\rho\phi \\ -\bar{q}_\rho\bar{\phi} & - \end{pmatrix}, \qquad 1 \leqslant \rho \leqslant s,$$

$$\pi_0 = \begin{pmatrix} \omega_{\bar{0}\bar{0}} & 0 \\ 0 & \omega_{0\bar{0}} \end{pmatrix}, \tag{3.20}$$

$$\pi_{-\rho} = \begin{pmatrix} - & q_{-\rho}\phi \\ -\bar{q}_{-\rho}\bar{\phi} & - \end{pmatrix}, \qquad 1 \leqslant \rho \leqslant s-1.$$

The theorem now follows easily from the structure equations of the unitary frame $\{\bar{Z}_{2s+1}, \bar{Z}_{2s}, \cdots, \bar{Z}_1, \bar{Z}_0, Z_0, Z_1, Z_2, \cdots, Z_{2s}, Z_{2s+1}\}$ using (3.12), (3.18), (3.19), (3.20), and Theorem 2.1. q.e.d.

The operation of passing from the Frenet harmonic sequence (3.8) to the Frenet harmonic sequence (3.17) is called *crossing*. Note that Theorem 3.2 ensures that the harmonic sequence obtained from (3.17) by adding $\partial\colon \lambda_{s+1} \to \lambda_{s+2}$ (or by adding $\bar{\partial}\colon \lambda_{-(s+1)} \to \lambda_{-(s+2)}$) is a Frenet harmonic sequence.

Theorem 3.3 shows that we must consider Frenet harmonic sequences of the form

$$\lambda_{-s} \xrightarrow{\partial} \lambda_{-(s-1)} \xrightarrow{\partial} \cdots \to \lambda_{-1} \xrightarrow{\partial} \lambda_1 \xrightarrow{\partial} \lambda_2 \xrightarrow{\partial} \cdots \xrightarrow{\partial} \lambda_s, \tag{3.21}$$

where $\lambda_{-\rho} = \bar{\lambda}_\rho$, $1 \leqslant \rho \leqslant s$. Consider the fundamental collineations $\partial_{-1}$: $\lambda_{-1} \to \lambda_1$ and $\bar{\partial}_1$: $\lambda_1 \to \lambda_{-1}$. By (3.2), $\sigma(\partial_{-1}) = \bar{\partial}_1$ and by (3.3), $\bar{\partial}_1 = -\sigma\ {}^t(\partial_{-1})$. Thus, $\partial_{-1} = -\ {}^t\partial_{-1}$, i.e., $\partial_{-1}$ is skew-symmetric. If $\{Z_0, Z_1\}$ is a unitary framing of $\lambda_1$, then with respect to the framings $\{\bar{Z}_1, \bar{Z}_0\}$ and $\{Z_0, Z_1\}$ of $\lambda_{-1}$ and $\lambda_1$, respectively, $\partial_{-1}$ has the form

$$\partial_{-1} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}.$$

With this observation the techniques discussed previous to Theorem 3.3 can be applied to the sequence (3.21). We have

**Theorem 3.4.** *The operation of crossing can be applied to the sequence* (3.21) *to construct a Frenet harmonic sequence*

$$l_{-s} \xrightarrow{\partial} l_{-s-1} \xrightarrow{\partial} \cdots l_{-1} \xrightarrow{\partial} l_0 \xrightarrow{\partial} l_1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} l_s, \tag{3.22}$$

*where* $\bar{l}_\rho = l_{-\rho}$, $0 \leqslant \rho \leqslant s$.

The process inverse to crossing, called *recrossing*, comes in two types: one inverting the construction of Theorem 3.3, the other inverting the construction of Theorem 3.4. The inversion of the construction of Theorem 3.3 involves an arbitrariness as it passes from a real Frenet harmonic sequence of the type of (3.17) to a real Frenet harmonic sequence of the type of (3.8). This operation proceeds as follows:

Consider the real Frenet harmonic sequence (3.17). The projective line $\lambda_1$ describes a $P_1$-bundle over $S^2$. $\lambda_1$ is a subbundle of the trivial $\mathbf{C}^n$-bundle over $S^2$. Using the standard connection on the trivial $\mathbf{C}^n$-bundle and the Newlander-Nirenberg theorem, $\lambda_1$ admits a natural holomorphic structure (cf. [4]). The arbitrariness of recrossing involves choosing a holomorphic section, $\sigma$, of the bundle $\lambda_1$. Adapt a unitary framing to (3.17) by setting

$$\begin{gathered} Z_1 = \sigma, \\ \underbrace{\partial \circ \cdots \circ \partial}_{\rho}(Z_1) = (\ )Z_{2\rho+1}, \qquad 1 \leqslant \rho \leqslant s, \end{gathered} \tag{3.23}$$

and letting $Z_{2\rho+2}$ be a point on $\lambda_{\rho+1}$ of unit length and orthogonal to $Z_{2\rho+1}$. As above $\{\bar{Z}_{2\rho+2}, \bar{Z}_{2\rho+1}\}$ will be a framing of the line $\lambda_{-(\rho+1)}$. Set

$$\begin{aligned} L_{-\rho} &= \bar{Z}_{2\rho+1} \wedge \bar{Z}_{2\rho}, \\ L_0 &= \bar{Z}_1 \wedge Z_1, \qquad 1 \leqslant \rho \leqslant s. \\ L_\rho &= Z_{2\rho} \wedge Z_{2\rho+1}, \end{aligned} \tag{3.24}$$

**Theorem 3.5.** *The lines* (3.24) *form a real Frenet harmonic sequence*

$$L_{-s} \xrightarrow{\partial} \cdots \rightarrow L_0 \xrightarrow{\partial} \cdots \xrightarrow{\partial} L_s. \tag{3.25}$$

*For appropriate choice of a holomorphic section,* $\sigma$, *the sequence* (3.25) *is the original sequence* (3.8).

On the other hand the inversion of the construction of Theorem 3.4 involves no arbitrariness. Consider the real Frenet harmonic sequence (3.22). The projective line $l_0$ admits a unitary frame $\{\overline{Z}_0, Z_0\}$, where the vector $Z_0$ is unique up to multiplication by a factor of absolute value 1. Set $\partial(\overline{Z}_0) = (\ )Z_1$ and adapt a unitary frame to (3.22) as in (3.23). Recrossing now proceeds as above.

We conclude that to invert the construction (by crossing) of the real Frenet harmonic sequence

$$l_{-(s+1)} \xrightarrow{\partial} \cdots \xrightarrow{\partial} l_0 \xrightarrow{\partial} \cdots \xrightarrow{\partial} l_{s+1}$$

from the real Frenet harmonic sequence

$$L_{-s} \xrightarrow{\partial} \cdots \xrightarrow{\partial} L_0 \xrightarrow{\partial} \cdots \xrightarrow{\partial} L_s$$

requires the choice of one holomorphic section of a $P_1$-bundle over $S^2$.

## 4. Main theorem

Given a harmonic map $L_0$: $S^2 \rightarrow G(2, n; \mathbf{R})$ and its associated real harmonic sequence, by successively applying crossing we can construct longer and longer real Frenet harmonic sequences. However, crossing cannot be applied to a real Frenet harmonic sequence

$$\lambda_{-s} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \lambda_0 \xrightarrow{\partial} \cdots \xrightarrow{\partial} \lambda_s, \tag{4.1}$$

when the fundamental collineation $\partial_s$: $\lambda_s \rightarrow \lambda_{s+1}$ is degenerate. There are two cases: (i) The fundamental collineation $\partial_s$: $\lambda_s \rightarrow \lambda_{s+1}$ is zero, or (ii) the fundamental collineation $\partial_s$: $\lambda_s \rightarrow \lambda_{s+1}$ has rank one. In case (i) the map $\lambda_{-s}$: $S^2 \rightarrow G(2, n; \mathbf{C})$ is holomorphic and the map $\lambda_s$: $S^2 \rightarrow G(2, n; \mathbf{C})$ is antiholomorphic. Moreover, for $0 \leqslant \rho \leqslant s - 1$, the $\rho$th osculating spaces of both $\lambda_{-s}$ and $\lambda_s$ are null. In case (ii) the harmonic maps $\lambda_{-s}$ and $\lambda_s$ are degenerate (i.e. one of their fundamental collineations is degenerate). Here, for $0 \leqslant \rho \leqslant s - 1$, the $\rho$th holomorphic (resp. antiholomorphic) osculating space of $\lambda_{-s}$ (resp. $\lambda_s$) is null. A degenerate harmonic map can be constructed from holomorphic curves $S^2 \rightarrow P_{n-1}$ via the procedure of *returning* (cf. [4]). In fact, if $\lambda_s$ is constructed from the curve $\delta$: $S^2 \to P_{n-1}$, then $\lambda_{-s} = \bar{\lambda}_s$ is constructed from the curve $\bar{\delta}$: $S^2 \to P_{n-1}$. By iterating crossing until the ambient space is exhausted, one of case (i) or (ii) must occur. We will call a holomorphic or an antiholomorphic curve $M \to G(2, n; \mathbf{C})$ *r-null* if its $\rho$th osculating spaces are null for $0 \leqslant \rho \leqslant r$. We will call a degenerate harmonic map $M \to G(2, n; \mathbf{C})$ *r-null* if its $\rho$th osculating spaces in the nondegenerate direction are null for $0 \leqslant \rho \leqslant r$. We have

**Theorem 4.1.** *Let $f$: $S^2 \to G(2, n$: $\mathbf{C})$ be a real nondegenerate harmonic map. Then there is associated to $f$ either* (i) *a unique r-null holomorphic curve $\Delta_f$: $S^2 \to G(2, n; \mathbf{C})$, $r \geqslant 0$, or* (ii) *a unique r-null degenerate harmonic map $\Delta_f$: $S^2 \to G(2, n; \mathbf{C})$, $r \geqslant 0$. $f$ can be recovered from $\Delta_f$ via recrossing and the $\partial$ and $\bar{\partial}$ transforms.*

The curve $\Delta_f$ is called the *directrix curve* of $f$. Since degenerate harmonic maps can be constructed from holomorphic curves $S^2 \to P_{n-1}$ using returnings, Theorem 4.1 provides a description of the harmonic maps $S^2 \to Q_{n-2}$.

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